

TRACE FORMULAS OF THE HECKE OPERATOR ON THE SPACES OF NEWFORMS

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1. INTRODUCTION

For a positive integer N and an even positive integer k , let $\mathcal{S}_k(N)$ be the space of all cuspforms of weight k with respect to the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \in N\mathbb{Z} \right\}$$

of level N , and $\mathcal{S}_k^0(N)$ the subspace of $\mathcal{S}_k(N)$ consisting of all newforms (cf. [3, Definition 5.6.1]). For a positive integer l , we denote by $T(l)$ the l -th Hecke operator on $\mathcal{S}_k(N)$. The first purpose of this paper is to write down on $\mathrm{tr}(T(l)|_{\mathcal{S}_k^0(N)})$, for square-free l and general N :

Theorem 1. *If $l > 1$ is square-free and $(l, N//l) = 1$, then we have*

$$\mathrm{tr}(T(l)|_{\mathcal{S}_k^0(N)}) = - \sum_{t \in \mathbb{T}(l)} a_{t,l,k} h_{t,l} \Lambda_{t,l}(N) + \delta_{k,2} \mu(N) \prod_{p \in \mathbb{P}(l//N)} (1+p).$$

Here $m//n = m/(m,n)$, $\mathbb{P}(n)$ is the set of all prime divisors of n , μ the Möbius function,

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$$

and $\mathbb{T}(l) = \mathbb{T}_-(l) \cup \mathbb{T}_\square(l)$ with

$$\mathbb{T}_-(l) = \{t \in \mathbb{Z} \mid t^2 - 4l < 0\},$$

$$\mathbb{T}_\square(l) = \{t \in \mathbb{Z} \mid t^2 - 4l \text{ is square}\}.$$

For each $t \in \mathbb{T}_-(l)$ (resp. $\mathbb{T}_\square(l)$), we put

$$a_{t,l,k} = \frac{\zeta^{k-1} - \eta^{k-1}}{\zeta - \eta} \left(\text{resp. } \frac{\min\{\zeta^{k-1}, \eta^{k-1}\}}{2|\zeta - \eta|} \right),$$

with ζ, η two roots of $X^2 - tX + l = 0$,

$$h_{t,l} = \frac{h(t^2 - 4l)}{\text{the number of units in } \mathbb{Q}(\sqrt{t^2 - 4l})} \quad (\text{resp. } 1).$$

For the definition of $\Lambda_{t,l}$, see §4.

Theorem 1 has been conjectured by Kazuhide Kubo [6], for N square-free, prime to 6, and $l|N$. We note that if $(l, N//l) \neq 1$ then $\mathrm{tr}(T(l)|_{\mathcal{S}_k^0(N)}) = 0$ (cf. [8, Theorem 4.6.17(3)]). Theorem 1 is derived from the Atkin-Lehner theory (cf [1] or [3, §5]) and the Eichler-Selberg trace formula ([8, Theorem 6.8.4] or [4]) on $T(l)|_{\mathcal{S}_k(N)}$. Some explicit calculations on $\Lambda_{t,l}$ is performed in §5 and we write down concrete examples of Theorem 1 for some l and $N \leq 42$ in §6.

For $f \in \mathcal{S}_k^0(N)$, we denote $a_n(f)$ the n -th Fourier coefficient of f . We say f is primitive if $f|T(l) = a_l(f)f$ for all positive integers l , and put $\mathcal{P}_k(N)$ the set of all primitive forms in $\mathcal{S}_k^0(N)$. We put $\varkappa = \frac{k}{2} - 1$ throughout this paper. With $N^\times = \prod_{p \in \mathbb{P}(N), p^2 \nmid N} p$, we define for each positive divisor i of N^\times ,

$$\mathcal{P}_k(N; i) = \{f \in \mathcal{P}_k(N) \mid p \in \mathbb{P}(i) \implies a_p(f) = -p^\varkappa, p \in \mathbb{P}(\frac{N^\times}{i}) \implies a_p(f) = p^\varkappa\},$$

$$\mathcal{S}_k^0(N; i) = \bigoplus_{f \in \mathcal{P}_k(N; i)} \mathbb{C}f.$$

Then we see $\mathcal{P}_k(N) = \sqcup_{i|N^\times} \mathcal{P}_k(N; i)$ (cf. Miyake[6, Theorem 4.6.17]), thus

$$\mathcal{S}_k^0(N) = \bigoplus_{f \in \mathcal{P}_k(N)} \mathbb{C}f = \bigoplus_{i|N^\times} \mathcal{S}_k^0(N; i),$$

and $\mathcal{S}_k^0(N; i)$ is a Hecke submodule of $\mathcal{S}_k^0(N)$ since so is $\mathbb{C}f$. We obtain

Theorem 2. *If $i|N^\times$ and $(l, N^\times) = 1$, then we have*

$$\mathrm{tr}(T(l)|_{\mathcal{S}_k^0(N; i)}) = \frac{1}{\sigma(N^\times)} \sum_{h|N^\times} \langle h, i \rangle h^{-\varkappa} \mathrm{tr}(T(hl)|_{\mathcal{S}_k^0(N)})$$

where $\sigma(n)$ is the number of all divisors of n and $\langle h, i \rangle = (-1)^{\#(\mathbb{P}(h) \cap \mathbb{P}(i))}$.

Theorem 2 has been conjectured also by Kubo [6], for N square-free, prime to 6 and $l = 1$. We give a proof of Theorem 2 in §7. Remark that

$$f \in \mathcal{P}_k(N; i), \nu \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \implies f^\nu \in \mathcal{P}_k(N; i)$$

where $f^\nu = \sum_{n \in \mathbb{N}} \nu(a_n(f))q^n$ ($q = e^{2\pi\sqrt{-1}z}$), since $f^\nu \in \mathcal{P}_k(N)$ and $\nu(\pm p^\varkappa) = \pm p^\varkappa$.

In section 8, as an application of Theorem 1 and 2, we calculate $\dim(\mathcal{S}_k^0(N; i))$ for $N \leq 42$. By virtue of our trace formulas, the Fourier coefficients of each primitive form may be calculated, in particular, we decide some primitive forms in terms of some Eisenstein series for $N = 14$, in the last section.

2. DIMENSION OF $\mathcal{S}_k^0(N)$

For reader's convenience, we review basic facts on arithmetic functions and show Martin's formula (cf [7, Theorem 1]).

For each $f, g : \mathbb{N} \rightarrow \mathbb{Q}$ we define the convolution product $f * g : \mathbb{N} \rightarrow \mathbb{Q}$ by

$$(f * g)(x) = \sum_{d|x} f\left(\frac{x}{d}\right)g(d).$$

Then we see the set of all functions $\mathbb{N} \rightarrow \mathbb{Q}$ is a ring under this product with $\delta = \delta_{\bullet, 1}$ unit element. We put $1(x) = 1$, then we see $1 * \mu = \delta$. We define

$$f[l](x) = \begin{cases} f(x) & \text{if } (x, l) = 1, \\ 0 & \text{if } (x, l) \neq 1, \end{cases}$$

then we see $f[l] * g[l] = (f * g)[l]$. We say that $f : \mathbb{N} \rightarrow \mathbb{Q}$ is multiplicative if $f(1) = 1$ and

$$(m, n) = 1 \implies f(mn) = f(m)f(n).$$

We see that $\delta, 1, \mu$ are multiplicative, and if f, g is multiplicative then so are $f * g, f[l]$. In particular, $\mu * \mu[l]$ is also multiplicative.

Lemma 3. *For each $n \geq 1$ and prime number p , we see*

$$(\mu * \mu)(p^n) = \begin{cases} -2 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$

in addition if $p|l$ then

$$(\mu * \mu[l])(p^n) = -\delta_{n,1}.$$

For each integer d , we denote by K_d the multiplicative function such that

$$K_d(p^n) = \begin{cases} \left(\frac{d}{p}\right) - 1 & \text{if } n = 1, \\ -\left(\frac{d}{p}\right) & \text{if } n = 2, p \nmid d, \\ -1 & \text{if } n = 2, p|d, \\ 1 & \text{if } n = 3, p|d, \\ 0 & \text{otherwise} \end{cases}$$

for each prime number p , where $\left(\frac{\cdot}{p}\right)$ is the Kronecker symbol.

For each $f \in \mathcal{S}_k(N)$ and $h \in \mathbb{N}$, we define $f^{(h)}(z) = f(hz)$. Then, we see

$$\mathcal{S}_k(N) = \bigoplus_{M|N} \bigoplus_{h|\frac{N}{M}} \mathcal{S}_k(M)^{(h)}$$

by the Atkin-Lehner theory, and thus

$$\dim \mathcal{S}_k(x) = \sum_{M|x} \sum_{h|\frac{x}{M}} \dim \mathcal{S}_k^0(M) = (1 * 1 * \dim \mathcal{S}_k^0)(x),$$

i.e., $\dim \mathcal{S}_k^0 = \mu * \mu * \dim \mathcal{S}_k$. On the other hand, we have

$$\dim \mathcal{S}_k = \frac{k-1}{12} \varepsilon_d + \frac{1}{4} (-1)^{\frac{k}{2}} \varepsilon_2 - \frac{1}{3} \left(\frac{k-1}{3}\right) \varepsilon_3 - \frac{1}{2} \varepsilon_\infty + \delta_{k,2} 1,$$

where $\varepsilon_d, \varepsilon_2, \varepsilon_3$ are the multiplicative functions satisfying

$$\varepsilon_d(p^n) = p^n + p^{n-1},$$

$$\varepsilon_2(2^n) = \delta_{n,1}, \quad \varepsilon_2(p^n) = 1 + \left(\frac{-1}{p}\right) \text{ if } p \neq 2,$$

$$\varepsilon_3(3^n) = \delta_{n,1}, \quad \varepsilon_3(p^n) = 1 + \left(\frac{-3}{p}\right) \text{ if } p \neq 3,$$

$$\varepsilon_\infty(p^n) = p^{\lfloor \frac{n}{2} \rfloor} + p^{\lfloor \frac{n-1}{2} \rfloor}$$

for each $n \geq 1$ and prime p (cf. [3, Theorem 3.5.1]). We note $\varepsilon_2(p^n) = 1 + \left(\frac{-4}{p}\right)$ if $p \neq 2$. It follows from Lemma 3 that

$$\dim \mathcal{S}_k^0 = \frac{k-1}{12} \mu * \mu * \varepsilon_d + \frac{1}{4} (-1)^{\frac{k}{2}} K_{-4} - \frac{1}{3} \left(\frac{k-1}{3}\right) K_{-3} - \frac{1}{2} \mu * \mu * \varepsilon_\infty + \delta_{k,2} \mu,$$

$$(\mu * \mu * \varepsilon_d)(p^n) = \begin{cases} p-1 & \text{if } n = 1, \\ p^2 - p - 1 & \text{if } n = 2, \\ p^{n-3}(p-1)(p^2-1) & \text{if } n \geq 3, \end{cases}$$

$$(\mu * \mu * \varepsilon_\infty)(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ p-2 & \text{if } n = 2, \\ p^{\frac{n}{2}-2}(p-1)^2 & \text{if } n \text{ is even, } \geq 4. \end{cases}$$

Example 4. For each $N \leq 42$, we write down the following formulas:

$$\begin{aligned}
\dim \mathcal{S}_k^0(1) &= \frac{1}{12}(k-1) + \frac{1}{4}(-1)^{\frac{k}{2}} - \frac{1}{3}\left(\frac{k-1}{3}\right) - \frac{1}{2} + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(2) &= \frac{1}{12}(k-1) - \frac{1}{4}(-1)^{\frac{k}{2}} + \frac{2}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(3) &= \frac{1}{6}(k-1) - \frac{1}{2}(-1)^{\frac{k}{2}} + \frac{1}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(4) &= \frac{1}{12}(k-1) - \frac{1}{4}(-1)^{\frac{k}{2}} - \frac{1}{3}\left(\frac{k-1}{3}\right), \\
\dim \mathcal{S}_k^0(5) &= \frac{1}{3}(k-1) + \frac{2}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(6) &= \frac{1}{6}(k-1) + \frac{1}{2}(-1)^{\frac{k}{2}} - \frac{2}{3}\left(\frac{k-1}{3}\right) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(7) &= \frac{1}{2}(k-1) - \frac{1}{2}(-1)^{\frac{k}{2}} - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(8) &= \frac{1}{4}(k-1) + \frac{1}{4}(-1)^{\frac{k}{2}}, \\
\dim \mathcal{S}_k^0(9) &= \frac{5}{12}(k-1) + \frac{1}{4}(-1)^{\frac{k}{2}} + \frac{1}{3}\left(\frac{k-1}{3}\right) - \frac{1}{2}, \\
\dim \mathcal{S}_k^0(10) &= \frac{1}{3}(k-1) - \frac{4}{3}\left(\frac{k-1}{3}\right) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(11) &= \frac{5}{6}(k-1) - \frac{1}{2}(-1)^{\frac{k}{2}} + \frac{2}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(12) &= \frac{1}{6}(k-1) + \frac{1}{2}(-1)^{\frac{k}{2}} + \frac{1}{3}\left(\frac{k-1}{3}\right), \\
\dim \mathcal{S}_k^0(13) &= (k-1) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(14) &= \frac{1}{2}(k-1) + \frac{1}{2}(-1)^{\frac{k}{2}} + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(15) &= \frac{2}{3}(k-1) - \frac{2}{3}\left(\frac{k-1}{3}\right) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(16) &= \frac{k}{2} - 1, \\
\dim \mathcal{S}_k^0(17) &= \frac{4}{3}(k-1) + \frac{2}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(18) &= \frac{5}{12}(k-1) - \frac{1}{4}(-1)^{\frac{k}{2}} - \frac{2}{3}\left(\frac{k-1}{3}\right), \\
\dim \mathcal{S}_k^0(19) &= \frac{3}{2}(k-1) - \frac{1}{2}(-1)^{\frac{k}{2}} - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(20) &= \frac{1}{3}(k-1) + \frac{2}{3}\left(\frac{k-1}{3}\right), \\
\dim \mathcal{S}_k^0(21) &= (k-1) + (-1)^{\frac{k}{2}} + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(22) &= \frac{5}{6}(k-1) + \frac{1}{2}(-1)^{\frac{k}{2}} - \frac{4}{3}\left(\frac{k-1}{3}\right) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(23) &= \frac{11}{6}(k-1) - \frac{1}{2}(-1)^{\frac{k}{2}} - \frac{2}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(24) &= \frac{1}{2}(k-1) - \frac{1}{2}(-1)^{\frac{k}{2}}, \\
\dim \mathcal{S}_k^0(25) &= \frac{19}{12}(k-1) - \frac{1}{4}(-1)^{\frac{k}{2}} - \frac{1}{3}\left(\frac{k-1}{3}\right) - \frac{3}{2}, \\
\dim \mathcal{S}_k^0(26) &= (k-1) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(27) &= \frac{4}{3}(k-1) - \frac{1}{3}\left(\frac{k-1}{3}\right), \\
\dim \mathcal{S}_k^0(28) &= \frac{1}{2}(k-1) + \frac{1}{2}(-1)^{\frac{k}{2}}, \\
\dim \mathcal{S}_k^0(29) &= \frac{7}{3}(k-1) + \frac{2}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(30) &= \frac{2}{3}(k-1) + \frac{4}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(31) &= \frac{5}{2}(k-1) - \frac{1}{2}(-1)^{\frac{k}{2}} - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(32) &= k-1,
\end{aligned}$$

$$\begin{aligned}
\dim \mathcal{S}_k^0(33) &= \frac{5}{3}(k-1) + (-1)^{\frac{k}{2}} - \frac{2}{3}\left(\frac{k-1}{3}\right) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(34) &= \frac{4}{3}(k-1) - \frac{4}{3}\left(\frac{k-1}{3}\right) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(35) &= 2(k-1) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(36) &= \frac{5}{12}(k-1) - \frac{1}{4}(-1)^{\frac{k}{2}} + \frac{1}{3}\left(\frac{k-1}{3}\right), \\
\dim \mathcal{S}_k^0(37) &= 3(k-1) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(38) &= \frac{3}{2}(k-1) + \frac{1}{2}(-1)^{\frac{k}{2}} + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(39) &= 2(k-1) + \delta_{k,2}, \\
\dim \mathcal{S}_k^0(40) &= k-1, \\
\dim \mathcal{S}_k^0(41) &= \frac{10}{3}(k-1) + \frac{2}{3}\left(\frac{k-1}{3}\right) - \delta_{k,2}, \\
\dim \mathcal{S}_k^0(42) &= (k-1) - (-1)^{\frac{k}{2}} - \delta_{k,2}.
\end{aligned}$$

3. RELATION LEMMA

We define the l -th Hecke operator $T_N(l)$ on $\mathcal{S}_k(N)$ by

$$f|T_N(l) = l^{k-1} \sum_{d|l, (l/d, N)=1} \sum_{b=0}^{d-1} d^{-k} f\left(\left(\frac{l}{d}z + b\right)/d\right)$$

(cf. [8, 4.5.26]). First, the following relations of Hecke operators on different levels are easily shown (cf. [3, in the proof of Proposition 5.6.2]):

Lemma 5. *Suppose that $f \in \mathcal{S}_k(N)$.*

(a) *If p, q are prime numbers with $p \neq q$, then we see*

$$\begin{aligned}
f|T_{pN}(q) &= f|T_N(q), \\
f^{(p)}|T_{pN}(q) &= (f|T_N(q))^{(p)}.
\end{aligned}$$

(b) *If p is a prime number with $(p, N) = 1$, then we see*

$$f^{(p)}|T_{pN}(p) = f.$$

For a divisor M of N and $f \in \mathcal{P}_k(M)$, we put

$$\mathcal{V}(f, N) = \bigoplus_{h|\frac{N}{M}} \mathbb{C} \cdot f^{(h)}.$$

Lemma 6. *Suppose $M|N$ and $f = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{P}_k(M)$. If l is square-free, then we have*

$$\mathrm{tr}(T_N(l)|_{\mathcal{V}(f, N)}) = (1 * I[l])\left(\frac{N}{M}\right)a_l.$$

Proof. For $h|\frac{N}{M}$, we show

$$f^{(h)}|T_N(l) = a_{l//h} f^{(h//l)}.$$

by induction on $\#\mathbb{P}(l)$. First, if $\#\mathbb{P}(l) = 0$, that is $l = 1$, then we get the assertion since $T_N(1)$ is the identity operator and $a_1 = 1$. Next, suppose the assertion is true for l and take a prime $p \nmid l$, then we see

$$f^{(h)}|T_N(p) = \begin{cases} a_p f^{(h)} & \text{if } p \nmid h, \\ f^{(h/p)} & \text{if } p|h \end{cases}$$

by Lemma 5, thus

$$f^{(h)}|T_N(lp) = a_{l//h} f^{(h//l)}|T_N(p) = a_{lp//h} f^{(h//lp)}.$$

Now, taking a basis $\{f^{(h)}\}_{h|\frac{N}{M}}$ of $\mathcal{V}(f, N)$, we see

$$\mathrm{tr}(T_N(l)|_{\mathcal{V}(f, N)}) = \sum_{h|\frac{N}{M}} \delta_{h, h//l} a_{l//h} = \sum_{h|\frac{N}{M}} I[l](h) a_l,$$

thus we get the assertion. \square

Lemma 7. *If l is square-free, then we have*

$$\mathrm{tr}_{k,l}^0 = \mu * \mu[l] * \mathrm{tr}_{k,l},$$

where we define the function $\mathrm{tr}_{k,l}$ by $x \mapsto \mathrm{tr}(T(l)|_{\mathcal{S}_k(x)})$ and $\mathrm{tr}_{k,l}^0$ by $x \mapsto \mathrm{tr}(T(l)|_{\mathcal{S}_k^0(x)})$.

Proof. By the above Lemma and the Atkin-Lehner theory, we have

$$\begin{aligned} \mathrm{tr}_{k,l}(x) &= \sum_{M|x} \sum_{f \in \mathcal{P}_k(M)} \mathrm{tr}(T_N(l)|_{\mathcal{V}(f, N)}) \\ &= \sum_{M|x} (1 * I[l])\left(\frac{N}{M}\right) \mathrm{tr}(T_M(l)|_{\mathcal{S}_k^0(M)}) \\ &= (1 * I[l] * \mathrm{tr}_{k,l}^0)(x). \end{aligned}$$

\square

4. PROOF OF THEOREM 1

For each $t \in \mathbb{T}(l)$, we put $d(t, l)$ the discriminant of $\mathbb{Q}(\sqrt{t^2 - 4l})$ and $m(t, l) = \sqrt{\frac{t^2 - 4l}{d(t, l)}}$. With $d = d(t, l)$ and $m = m(t, l)$, for each $\phi|m$, we define

$$b_{t,l,\phi} = \phi \prod_{p \in \mathbb{P}(\phi)} \left(1 - \left(\frac{d}{p}\right)p^{-1}\right),$$

and the multiplicative function $c_{t,l,\phi}$ by

$$c_{t,l,\phi}(p^n) = \sum_{\xi \in \Omega/\psi p^n} \mathbf{1}_p(\xi) + \sum_{\xi \in \Omega'/\psi p^n} \mathbf{1}_p(t - \xi)$$

for each $n \geq 1$ and prime p , where $\psi = \frac{m}{\phi}$, $\mathbf{1}_p$ is the trivial character mod p and

$$\Omega = \{\xi \in \mathbb{Z}_p \mid \xi^2 - t\xi + l \equiv 0 \pmod{\psi^2 p^n \mathbb{Z}_p}\},$$

$$\Omega' = \{\xi \in \mathbb{Z}_p \mid \xi^2 - t\xi + l \equiv 0 \pmod{\psi^2 p^{n+1} \mathbb{Z}_p}\} \text{ if } p|d\phi, = \emptyset \text{ otherwise.}$$

We note that if $t \in \mathbb{T}_{\square}(l)$ then $d = 1$ and $b_{t,l,\phi} = \#(\mathbb{Z}/\phi\mathbb{Z})^\times$.

Suppose that $l > 1$ is square-free and define the multiplicative function ω_l by

$$\omega_l(p^n) = \begin{cases} \frac{p}{1+p} & \text{if } p \in \mathbb{P}(l), \\ 1 & \text{if } p \notin \mathbb{P}(l). \end{cases}$$

The Eichler-Selberg trace formula says

$$\mathrm{tr}_{k,l} = - \sum_{t \in \mathbb{T}(l)} a_{t,l,k} h_{t,l} \sum_{\phi|m(t,l)} b_{t,l,\phi} c_{t,l,\phi} + \delta_{k,2} \prod_{p \in \mathbb{P}(l)} (1+p) \cdot \omega_l.$$

Now, we define

$$\Lambda_{t,l} = \sum_{\phi|m(t,l)} b_{t,l,\phi} (\mu * \mu[l] * c_{t,l,\phi}),$$

then we have

$$\mathrm{tr}_{k,l}^0 = - \sum_{t \in \mathbb{T}(l)} a_{t,l,k} h_{t,l} \Lambda_{t,l} + \delta_{k,2} \prod_{p \in \mathbb{P}(l)} (1+p) \cdot (\mu * \mu[l] * \omega_l).$$

and

$$\begin{aligned} (\mu * \mu[l] * \omega_l)(p) &= -\frac{1}{1+p} & \text{if } p \in \mathbb{P}(l), \\ (\mu * \mu[l] * \omega_l)(p^n) &= -\delta_{n,1} & \text{if } p \notin \mathbb{P}(l). \end{aligned}$$

If $(l, N//l) = 1$, then we obtain

$$\mathrm{tr}_{k,l}^0(N) = - \sum_{t \in \mathbb{T}(l)} a_{t,l,k} h_{t,l} \Lambda_{t,l}(N) + \delta_{k,2} \mu(N) \prod_{p \in \mathbb{P}(l//N)} (1+p).$$

5. CALCULATIONS ON Λ

Proposition 8. *Suppose that $l > 1$ is square-free, $(l, N//l) = 1$ and $(l, N) \nmid t$. Then we have*

$$\Lambda_{t,l}(N) = 0.$$

Proof. We see that there exists a prime number p such that $p|(l, N)$ and $p \nmid t$. We note $p^2 \nmid N$ since $(l, N//l) = 1$. We show for each $\phi|m(t, l)$, $(\mu * \mu[l] * c_{t,l,\phi})(p) = 0$. Indeed, we see $p \nmid (t^2 - 4l)$ and

$$\Omega = \{\xi \in \mathbb{Z}_p \mid \xi^2 - t\xi \equiv 0 \pmod{p\mathbb{Z}_p}\} = \{\xi \in \mathbb{Z}_p \mid \xi \equiv 0, t \pmod{p\mathbb{Z}_p}\},$$

$\Omega' = \emptyset$, hence $c_{t,l,\phi}(p) = \mathbf{1}_p(0) + \mathbf{1}_p(t) = 1$. We get the assertion by Lemma 3. \square

Proposition 9. *Suppose that $l > 1$ is square-free and $(l, N)|t$. Put $d = d(t, l)$ and $m = m(t, l)$. If $m = 1$, then we have*

$$\Lambda_{t,l}(N) = K_d(N).$$

If m is prime, then putting $v_m(N) = \max\{n \mid N \in m^n\mathbb{Z}\}$ we have

$$\Lambda_{t,l}(N) = K_d(Nm^{-v_m(N)}) \times \begin{cases} m+1 - (\frac{d}{m}) & \text{if } v_m(N) = 0, \\ (\frac{d}{m}) - 1 & \text{if } v_m(N) = 1, \\ m^2 - 2m - 1 + (\frac{d}{m}) & \text{if } v_m(N) = 2, \\ (m - (\frac{d}{m}))(m-1)((\frac{d}{m}) - 1) & \text{if } v_m(N) = 3, m \nmid d, \\ -(m - (\frac{d}{m}))m(\frac{d}{m}) & \text{if } v_m(N) = 4, m \nmid d, \\ 1 - m^2 & \text{if } v_m(N) = 3, m|d, \\ m(1-m) & \text{if } v_m(N) = 4, m|d, \\ m^2 & \text{if } v_m(N) = 5, m|d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For each $p|(l, N)$ and $\phi|m$, we see $c_{t,l,\phi}(p) = 0$ since

$$\Omega \subset \{\xi \in \mathbb{Z}_p \mid \xi^2 \equiv 0 \pmod{p\mathbb{Z}_p}\} = p\mathbb{Z}_p,$$

$$\Omega' \subset \{\xi \in \mathbb{Z}_p \mid \xi^2 - t\xi + l \equiv 0 \pmod{p^2\mathbb{Z}_p}\} = \emptyset,$$

and thus $(\mu * \mu[l] * c_{t,l,\phi})(p) = -1$ by Lemma 3. Therefore, the first assertion follows from Lemma 10 below and

$$\Lambda_{t,l} = \mu * \mu[l] * c_{t,l,1}.$$

The second one also follows from Lemmas 10, 11 below and

$$\Lambda_{t,l} = \mu * \mu[l] * c_{t,l,1} + (m - (\frac{d}{m}))\mu * \mu[l] * c_{t,l,m}.$$

□

In Lemmas 10 and 11, assume that $l > 1$ is square-free, $t \in \mathbb{T}(l)$, $n \geq 1$, and p is a prime number, and put $d = d(t, l)$ and $m = m(t, l)$. We note $d \equiv 0, 1 \pmod{4}$ and $t^2 - 4l = dm^2$.

Lemma 10. *If $p \nmid lm$, then for each $\phi|m$, we have*

$$(\mu * \mu[l] * c_{t,l,\phi})(p^n) = K_d(p^n).$$

Proof. We show

$$c_{t,l,\phi}(p^n) = \begin{cases} 1 + (\frac{d}{p}) & \text{if } p \nmid d, \\ \delta_{n,1} & \text{if } p|d, \end{cases}$$

then we get the assertion by Lemma 3. First, suppose $p \nmid d$. If $p \neq 2$, then we see

$$c_{t,l,\phi}(p^n) = \#(\{\xi \in \mathbb{Z}_p \mid (\xi - \frac{t}{2})^2 \equiv \frac{dm^2}{4} \pmod{p^n}\} / p^n) = 1 + (\frac{d}{p}).$$

If $p = 2$, then we see $d \equiv dm^2 \equiv t^2 - 4 \equiv 5 \pmod{8}$ and

$$c_{t,l,1}(2) = \#\{\xi \in \mathbb{Z}_2 \mid \xi^2 - \xi + 1 \equiv 0 \pmod{2}\} / 2 = 0 = 1 + (\frac{d}{2}).$$

We easily see $c_{t,l,1}(2^n) = 0$ for $n \geq 2$. Next, suppose $p|d$. If $p \neq 2$, then we see

$$\begin{aligned} c_{t,l,1}(p) &= \#(\{\xi \in \mathbb{Z}_p \mid (\xi - \frac{t}{2})^2 \equiv 0 \pmod{p}\} / p) \\ &\quad + \#(\{\xi \in \mathbb{Z}_p \mid (\xi - \frac{t}{2})^2 \equiv \frac{dm^2}{4} \pmod{p^2}\} / p) \\ &= 1 + 0, \end{aligned}$$

and $c_{t,l,1}(p^n) = 0$ for $n \geq 2$. If $p = 2$, then we see $2|t$, $\frac{d}{4} \equiv 2, 3 \pmod{4}$ and

$$\begin{aligned} c_{t,l,1}(2) &= \#(\{\xi \in \mathbb{Z}_2 \mid \xi^2 + 1 \equiv 0 \pmod{2}\} / 2) \\ &\quad + \#(\{\xi \in \mathbb{Z}_2 \mid (\xi - \frac{t}{2})^2 \equiv \frac{dm^2}{4} \pmod{4}\} / 2) \\ &= 1 + 0, \end{aligned}$$

and $c_{t,l,1}(2^n) = 0$ for $n \geq 2$. □

Lemma 11. *If m is prime and $m \nmid l$, then we have*

$$(\mu * \mu[l] * c_{t,l,1})(m^n) = K_d(m^n),$$

and

$$(\mu * \mu[l] * c_{t,l,m})(m^n) = \begin{cases} 0 & \text{if } n = 1, \\ m - 2 + (\frac{d}{m}) & \text{if } n = 2, \\ (m - 1)((\frac{d}{m}) - 1) & \text{if } n = 3, m \nmid d, \\ -m(\frac{d}{m}) & \text{if } n = 4, m \nmid d, \\ -m & \text{if } n = 3, m|d, \\ 1 - m & \text{if } n = 4, m|d, \\ m & \text{if } n = 5, m|d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The case $m = 2$: We note $2|t$. If $2 \nmid d$, then we see $4|t$,

$$c_{t,l,1}(2^n) = \#(\{\xi \in \mathbb{Z}_2 \mid (\xi - \frac{t}{2})^2 \equiv d \pmod{2^{n+2}}\} / 2^{n+1}) = 1 + (\frac{d}{2}),$$

and

$$\begin{aligned} c_{t,l,2}(2^n) &= \#(\{\xi \in \mathbb{Z}_2 \mid (\xi - \frac{t}{2})^2 \equiv d \pmod{2^n}\}/2^n) \\ &\quad + \#(\{\xi \in \mathbb{Z}_2 \mid (\xi - \frac{t}{2})^2 \equiv d \pmod{2^{n+1}}\}/2^n) \\ &= \begin{cases} 2 & \text{if } n = 1, \\ 3 + (\frac{d}{2}) & \text{if } n = 2, \\ 3(1 + (\frac{d}{2})) & \text{if } n \geq 3. \end{cases} \end{aligned}$$

If $2 \nmid d$, then we see $4 \nmid t$, $\frac{d}{4} \equiv 2, 3 \pmod{4}$,

$$\begin{aligned} c_{t,l,1}(2^n) &= \#(\{\xi \in \mathbb{Z}_2 \mid (\xi - \frac{t}{2})^2 \equiv d \pmod{2^{n+2}}\}/2^{n+1}) \\ &\quad + \#(\{\xi \in \mathbb{Z}_2 \mid (\xi - \frac{t}{2})^2 \equiv d \pmod{2^{n+3}}\}/2^{n+1}) \\ &= \delta_{n,1}, \end{aligned}$$

and

$$\begin{aligned} c_{t,l,2}(2^n) &= \#(\{\xi \in \mathbb{Z}_2 \mid (\xi - \frac{t}{2})^2 \equiv d \pmod{2^n}\}/2^n) \\ &\quad + \#(\{\xi \in \mathbb{Z}_2 \mid (\xi - \frac{t}{2})^2 \equiv d \pmod{2^{n+1}}\}/2^n) \\ &= \begin{cases} 2 & \text{if } n = 1, 3, \\ 3 & \text{if } n = 2, \\ 0 & \text{if } n \geq 4. \end{cases} \end{aligned}$$

The case $m \neq 2$: If $m \nmid d$, then we see

$$\begin{aligned} c_{t,l,1}(m^n) &= \#(\{\xi \in \mathbb{Z}_m \mid (\xi - \frac{t}{2})^2 \equiv \frac{d}{4} \pmod{m^n}\}/m^n) = 1 + (\frac{d}{2}), \\ c_{t,l,m}(m^n) &= \#(\{\xi \in \mathbb{Z}_m \mid (\xi - \frac{t}{2})^2 \equiv \frac{dm^2}{4} \pmod{m^n}\}/m^n) \\ &\quad + \#(\{\xi \in \mathbb{Z}_m \mid (\xi - \frac{t}{2})^2 \equiv \frac{dm^2}{4} \pmod{m^{n+1}}\}/m^n) \\ &= \begin{cases} 2 & \text{if } n = 1, \\ m + 1 + (\frac{d}{m}) & \text{if } n = 2, \\ (m + 1)(1 + (\frac{d}{m})) & \text{if } n \geq 3. \end{cases} \end{aligned}$$

If $m \mid d$, then we see

$$\begin{aligned} c_{t,l,1}(m^n) &= \#(\{\xi \in \mathbb{Z}_m \mid (\xi - \frac{t}{2})^2 \equiv \frac{d}{4} \pmod{m^n}\}/m^n) \\ &\quad + \#(\{\xi \in \mathbb{Z}_m \mid (\xi - \frac{t}{2})^2 \equiv \frac{d}{4} \pmod{m^{n+1}}\}/m^n) \\ &= \delta_{n,1}, \end{aligned}$$

$$\begin{aligned} c_{t,l,m}(m^n) &= \#(\{\xi \in \mathbb{Z}_m \mid (\xi - \frac{t}{2})^2 \equiv \frac{dm^2}{4} \pmod{m^n}\}/m^n) \\ &\quad + \#(\{\xi \in \mathbb{Z}_m \mid (\xi - \frac{t}{2})^2 \equiv \frac{dm^2}{4} \pmod{m^{n+1}}\}/m^n) \\ &= \begin{cases} 2 & \text{if } n = 1, \\ m + 1 & \text{if } n = 2, \\ m & \text{if } n = 3, \\ 0 & \text{if } n \geq 4. \end{cases} \end{aligned}$$

We get all the assertions by Lemma 3. □

6. EXAMPLES OF TRACE ON $\mathcal{S}_k^0(N)$

In this section, we give some concrete formulas. We first note $a_{-t,k,l} = a_{t,k,l}$, $h_{-t,l} = h_{t,l}$, $\Lambda_{-t,l} = \Lambda_{t,l}$, and

$$a_{0,l,k} = \frac{(-l)^{\frac{k-1}{2}} - (-l)^{\frac{k-1}{2}}}{(-l)^{\frac{1}{2}} - (-l)^{\frac{1}{2}}} = \frac{(-l)^{\varkappa} - (-l)^{\varkappa}}{1 - (-1)} = (-l)^{\varkappa}.$$

Lemma 12. *If l is prime, then we have $\mathbb{T}_{\square}(l) = \{\pm(l+1)\}$, $m(l+1, l) = l-1$ and $a_{l+1,l,k} = \frac{1}{2(l-1)}$.*

Proof. If $t^2 - 4l = m^2$, then we see $(t+m)(t-m) = 4l$, thus $t = l+1$, $m = l-1$ and $X^2 - tX + l = (X-1)(X-l)$. \square

Proposition 13. *If $(2, N//2) = 1$ i.e. $4 \nmid N$, then we get*

$$\begin{aligned} \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(N)}) &= -\frac{1}{2}(-2)^{\varkappa} (K_{-2}(N) + A_k K_{-4}(N)) - a_{1,2,k} K_{-7}(N) \\ &\quad - K_1(N) + \delta_{k,2} \mu(N) \prod_{p \in \mathbb{P}(2//N)} (1+p), \end{aligned}$$

where

$$A_k = \begin{cases} 1 & \text{if } k \equiv 0, 2 \pmod{8}, \\ -1 & \text{if } k \equiv 4, 6 \pmod{8}. \end{cases}$$

Proof. We see $\mathbb{T}_{-}(2) = \{0, \pm 1, \pm 2\}$, $h_{0,2} = h_{1,2} = \frac{1}{2}$, $h_{2,2} = \frac{1}{4}$, and

$$\begin{aligned} a_{2,2,k} &= \frac{(1+\sqrt{-1})^{k-1} - (1-\sqrt{-1})^{k-1}}{2\sqrt{-1}} \\ &= \frac{(\sqrt{-2})^{k-1}}{2\sqrt{-1}} \left(\left(\frac{1+\sqrt{-1}}{\sqrt{2}} \right)^{k-1} - \left(\frac{-1+\sqrt{-1}}{\sqrt{2}} \right)^{k-1} \right) \\ &= (-2)^{\varkappa} A_k. \end{aligned}$$

\square

$$\text{We note } a_{1,2,k} = \frac{\left(\frac{1+\sqrt{-7}}{2} \right)^{k-1} - \left(\frac{1-\sqrt{-7}}{2} \right)^{k-1}}{\sqrt{-7}}.$$

Example 14. *We write down the following formulas:*

$$\begin{aligned} \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(1)}) &= -\frac{1}{2}(-2)^{\varkappa} (1 + A_k) - a_{1,2,k} - 1 + 3\delta_{k,2}, \\ \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(2)}) &= \frac{1}{2}(-2)^{\varkappa} (1 + A_k) - \delta_{k,2}, \\ \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(3)}) &= (-2)^{\varkappa} A_k + 2a_{1,2,k} - 3\delta_{k,2}, \\ \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(5)}) &= (-2)^{\varkappa} + 2a_{1,2,k} - 3\delta_{k,2}, \\ \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(6)}) &= -(-2)^{\varkappa} A_k + \delta_{k,2}, \\ \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(7)}) &= (-2)^{\varkappa} (1 + A_k) + a_{1,2,k} - 3\delta_{k,2}, \\ \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(9)}) &= \frac{1}{2}(-2)^{\varkappa} (1 - A_k) - a_{1,2,k} + 1, \\ \text{tr}(\mathbb{T}(2)|_{\mathcal{S}_k^0(10)}) &= -(-2)^{\varkappa} + \delta_{k,2}, \end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(T(2)|_{S_k^0(11)}) &= (-2)^\kappa A_k - 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(13)}) &= (-2)^\kappa + 2a_{1,2,k} - 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(14)}) &= -(-2)^\kappa(1 + A_k) + \delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(15)}) &= -4a_{1,2,k} + 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(17)}) &= 2a_{1,2,k} - 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(18)}) &= -\frac{1}{2}(-2)^\kappa(1 - A_k), \\
\mathrm{tr}(T(2)|_{S_k^0(19)}) &= (-2)^\kappa A_k + 2a_{1,2,k} - 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(21)}) &= -2(-2)^\kappa A_k - 2a_{1,2,k} + 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(22)}) &= -(-2)^\kappa A_k + \delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(23)}) &= (-2)^\kappa(1 + A_k) - 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(25)}) &= -\frac{1}{2}(-2)^\kappa(1 - A_k) - a_{1,2,k} + 1, \\
\mathrm{tr}(T(2)|_{S_k^0(26)}) &= -(-2)^\kappa + \delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(27)}) &= 0, \\
\mathrm{tr}(T(2)|_{S_k^0(29)}) &= (-2)^\kappa - 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(30)}) &= -\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(31)}) &= (-2)^\kappa(1 + A_k) + 2a_{1,2,k} - 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(33)}) &= -2(-2)^\kappa A_k + 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(34)}) &= \delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(35)}) &= -2(-2)^\kappa - 2a_{1,2,k} + 3\delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(38)}) &= -(-2)^\kappa A_k + \delta_{k,2}, \\
\mathrm{tr}(T(2)|_{S_k^0(42)}) &= 2(-2)^\kappa A_k - \delta_{k,2}.
\end{aligned}$$

Proposition 15. *If $(3, N//3) = 1$ i.e. $9 \nmid N$, then we get*

$$\begin{aligned}
\mathrm{tr}(T(3)|_{S_k^0(N)}) &= -\frac{1}{6}(-3)^\kappa(\Lambda_{0,3}(N) + 2B_k K_{-3}(N)) - a_{1,3,k} K_{-11}(N) - a_{2,3,k} K_{-2}(N) \\
&\quad - \frac{1}{2}\Lambda_{4,3}(N) + \delta_{k,2}\mu(N) \prod_{p \in \mathbb{P}(3//N)} (1 + p),
\end{aligned}$$

where

$$B_k = \begin{cases} 1 & \text{if } k \equiv 0, 2 \pmod{6}, \\ -2 & \text{if } k \equiv 4 \pmod{6}. \end{cases}$$

In addition, we see

$$\begin{aligned}
\Lambda_{0,3}(N) &= K_{-3}(N2^{-v_2(N)}) \times \begin{cases} 4 & \text{if } v_2(N) = 0, \\ -2 & \text{if } v_2(N) = 1, 2, \\ -6 & \text{if } v_2(N) = 3, \\ 6 & \text{if } v_2(N) = 4, \\ 0 & \text{if } v_2(N) \geq 5, \end{cases} \\
\Lambda_{4,3}(N) &= K_1(N2^{-v_2(N)}) \times \begin{cases} 2 & \text{if } v_2(N) = 0, \\ -2 & \text{if } v_2(N) = 4, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. We see $\mathbb{T}_-(2) = \{0, \pm 1, \pm 2, \pm 3\}$, $h_{0,3} = h_{3,3} = \frac{1}{6}$, $h_{1,3} = h_{2,3} = \frac{1}{2}$, and

$$\begin{aligned} a_{3,3,k} &= \frac{\left(\frac{3+\sqrt{-3}}{2}\right)^{k-1} - \left(\frac{3-\sqrt{-3}}{2}\right)^{k-1}}{\sqrt{-3}} \\ &= \frac{(\sqrt{-3})^{k-1}}{\sqrt{-3}} \left(\left(\frac{1+\sqrt{-3}}{2}\right)^{k-1} - \left(\frac{-1-\sqrt{-3}}{2}\right)^{k-1} \right) \\ &= B_k(-3)^\varkappa. \end{aligned}$$

□

$$\text{We note } a_{1,3,k} = \frac{\left(\frac{1+\sqrt{-11}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{-11}}{2}\right)^{k-1}}{\sqrt{-11}} \text{ and } a_{2,3,k} = \frac{(1+\sqrt{-2})^{k-1} - (1-\sqrt{-2})^{k-1}}{2\sqrt{-2}}.$$

Example 16. We write down the following formulas:

$$\begin{aligned} \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(1)}) &= -\frac{1}{3}(-3)^\varkappa(2 + B_k) - a_{1,3,k} - a_{2,3,k} - 1 + 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(2)}) &= \frac{1}{3}(-3)^\varkappa(1 + 2B_k) + 2a_{1,3,k} + a_{2,3,k} - 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(3)}) &= \frac{1}{3}(-3)^\varkappa(2 + B_k) - \delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(4)}) &= \frac{1}{3}(-3)^\varkappa(1 - B_k) - a_{1,3,k} + a_{2,3,k}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(5)}) &= \frac{2}{3}(-3)^\varkappa(2 + B_k) + 2a_{2,3,k} - 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(6)}) &= -\frac{1}{3}(-3)^\varkappa(1 + 2B_k) + \delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(7)}) &= 2a_{1,3,k} + 2a_{2,3,k} - 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(8)}) &= (-3)^\varkappa - a_{2,3,k}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(10)}) &= -\frac{2}{3}(-3)^\varkappa(1 + 2B_k) - 2a_{2,3,k} + 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(11)}) &= \frac{2}{3}(-3)^\varkappa(2 + B_k) + a_{1,3,k} + 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(12)}) &= -\frac{1}{3}(-3)^\varkappa(1 - B_k), \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(13)}) &= 2a_{1,3,k} + 2a_{2,3,k} - 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(14)}) &= -4a_{1,3,k} - 2a_{2,3,k} + 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(15)}) &= -\frac{2}{3}(-3)^\varkappa(2 + B_k) + \delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(16)}) &= -(-3)^\varkappa + 1, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(17)}) &= \frac{2}{3}(-3)^\varkappa(2 + B_k) + 2a_{1,3,k} - 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(19)}) &= 2a_{1,3,k} - 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(20)}) &= -\frac{2}{3}(-3)^\varkappa(1 - B_k) - 2a_{2,3,k}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(21)}) &= \delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(22)}) &= -\frac{2}{3}(-3)^\varkappa(1 + 2B_k) - 2a_{1,3,k} + 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(23)}) &= \frac{2}{3}(-3)^\varkappa(2 + B_k) + 2a_{2,3,k} - 4\delta_{k,2}, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(24)}) &= -(-3)^\varkappa, \\ \text{tr}(\mathbf{T}(3)|_{\mathcal{S}_k^0(25)}) &= -\frac{1}{3}(-3)^\varkappa(2 + B_k) + a_{1,3,k} - a_{2,3,k} + 1, \end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(T(3)|_{S_k^0(26)}) &= -4a_{1,3,k} - 2a_{2,3,k} + 4\delta_{k,2}, \\
\mathrm{tr}(T(3)|_{S_k^0(28)}) &= 2a_{1,3,k} - 2a_{2,3,k}, \\
\mathrm{tr}(T(3)|_{S_k^0(29)}) &= \frac{2}{3}(-3)^\varkappa(2 + B_k) + 2a_{1,3,k} + 2a_{2,3,k} - 4\delta_{k,2}, \\
\mathrm{tr}(T(3)|_{S_k^0(30)}) &= \frac{2}{3}(-3)^\varkappa(1 + 2B_k) - \delta_{k,2}, \\
\mathrm{tr}(T(3)|_{S_k^0(31)}) &= 2a_{2,3,k} - 4\delta_{k,2}, \\
\mathrm{tr}(T(3)|_{S_k^0(32)}) &= 0, \\
\mathrm{tr}(T(3)|_{S_k^0(33)}) &= -\frac{2}{3}(-3)^\varkappa(2 + B_k) + \delta_{k,2}, \\
\mathrm{tr}(T(3)|_{S_k^0(34)}) &= -\frac{2}{3}(-3)^\varkappa(1 + 2B_k) - 4a_{1,3,k} + 4\delta_{k,2}, \\
\mathrm{tr}(T(3)|_{S_k^0(35)}) &= -4a_{2,3,k} + 4\delta_{k,2}, \\
\mathrm{tr}(T(3)|_{S_k^0(39)}) &= \delta_{k,2}, \\
\mathrm{tr}(T(3)|_{S_k^0(42)}) &= -\delta_{k,2}.
\end{aligned}$$

Lemma 17. *If $l > 1$ is square-free and $(l, t) > 1$, then we have $t \notin \mathbb{T}_\square(l)$.*

Proof. If $(l, t) > 2$, then there exists $p \in \mathbb{P}((l, t) \setminus \{2\})$ and $t^2 - 4l$ is not square since $v_p(t^2 - 4l) = 1$. If $(l, t) = 2$, then we see $l \equiv 2 \pmod{4}$ and $t^2 - 4l$ is not square since $(\frac{t}{2})^2 - l \equiv -2$ or $-1 \pmod{4}$. \square

Proposition 18. *If $l > 1$ is square-free, $(l, N//l) = 1$ and $(l, N) > 2\sqrt{l}$, then we have*

$$\mathrm{tr}(T(l)|_{S_k^0(N)}) = -\frac{h(-l)}{2}(-l)^\varkappa\Lambda_{0,l}(N) + \delta_{k,2}\mu(N) \prod_{p \in \mathbb{P}(l//N)} (1 + p).$$

Proof. By Theorem 1, Proposition 8 and Lemma 17, we see

$$\mathrm{tr}(T(l)|_{S_k^0(N)}) = -h_{0,l}(-l)^\varkappa\Lambda_{0,l}(N) + \delta_{k,2}\mu(N) \prod_{p \in \mathbb{P}(l//N)} (1 + p).$$

We note $l \geq 5$ since $l \geq (l, N) > 2\sqrt{l}$, thus $h_{0,l} = \frac{h(-l)}{2}$. \square

Example 19. *For each $l, N \leq 42$ satisfying the conditions of the above Proposition, we write down the following formulas:*

$$\begin{aligned}
\mathrm{tr}(T(5)|_{S_k^0(5)}) &= (-5)^\varkappa - \delta_{k,2}, \\
\mathrm{tr}(T(5)|_{S_k^0(10)}) &= -(-5)^\varkappa + \delta_{k,2}, \\
\mathrm{tr}(T(5)|_{S_k^0(15)}) &= \delta_{k,2}, \\
\mathrm{tr}(T(5)|_{S_k^0(20)}) &= -(-5)^\varkappa, \\
\mathrm{tr}(T(5)|_{S_k^0(30)}) &= -\delta_{k,2}, \\
\mathrm{tr}(T(5)|_{S_k^0(35)}) &= \delta_{k,2}, \\
\mathrm{tr}(T(5)|_{S_k^0(40)}) &= (-5)^\varkappa, \\
\\
\mathrm{tr}(T(6)|_{S_k^0(6)}) &= -(-6)^\varkappa + \delta_{k,2}, \\
\mathrm{tr}(T(6)|_{S_k^0(30)}) &= -\delta_{k,2}, \\
\mathrm{tr}(T(6)|_{S_k^0(42)}) &= -\delta_{k,2},
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(T(7)|_{\mathcal{S}_k^0(7)}) &= (-7)^{\mathcal{Z}} - \delta_{k,2}, \\
\mathrm{tr}(T(7)|_{\mathcal{S}_k^0(14)}) &= \delta_{k,2}, \\
\mathrm{tr}(T(7)|_{\mathcal{S}_k^0(21)}) &= -2(-7)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(7)|_{\mathcal{S}_k^0(28)}) &= 0, \\
\mathrm{tr}(T(7)|_{\mathcal{S}_k^0(35)}) &= -2(-7)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(7)|_{\mathcal{S}_k^0(42)}) &= -\delta_{k,2}, \\
\\
\mathrm{tr}(T(10)|_{\mathcal{S}_k^0(10)}) &= -(-10)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(10)|_{\mathcal{S}_k^0(30)}) &= 2(-10)^{\mathcal{Z}} - \delta_{k,2}, \\
\\
\mathrm{tr}(T(11)|_{\mathcal{S}_k^0(11)}) &= 2(-11)^{\mathcal{Z}} - \delta_{k,2}, \\
\mathrm{tr}(T(11)|_{\mathcal{S}_k^0(22)}) &= -(-11)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(11)|_{\mathcal{S}_k^0(33)}) &= \delta_{k,2}, \\
\\
\mathrm{tr}(T(13)|_{\mathcal{S}_k^0(13)}) &= (-13)^{\mathcal{Z}} - \delta_{k,2}, \\
\mathrm{tr}(T(13)|_{\mathcal{S}_k^0(26)}) &= -(-13)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(13)|_{\mathcal{S}_k^0(39)}) &= -2(-13)^{\mathcal{Z}} + \delta_{k,2}, \\
\\
\mathrm{tr}(T(14)|_{\mathcal{S}_k^0(14)}) &= -2(-14)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(14)|_{\mathcal{S}_k^0(42)}) &= -\delta_{k,2}, \\
\\
\mathrm{tr}(T(15)|_{\mathcal{S}_k^0(15)}) &= -2(-15)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(15)|_{\mathcal{S}_k^0(30)}) &= -\delta_{k,2}, \\
\\
\mathrm{tr}(T(17)|_{\mathcal{S}_k^0(17)}) &= 2(-17)^{\mathcal{Z}} - \delta_{k,2}, \\
\mathrm{tr}(T(17)|_{\mathcal{S}_k^0(34)}) &= -2(-17)^{\mathcal{Z}} + \delta_{k,2}, \\
\\
\mathrm{tr}(T(19)|_{\mathcal{S}_k^0(19)}) &= 2(-19)^{\mathcal{Z}} - \delta_{k,2}, \\
\mathrm{tr}(T(19)|_{\mathcal{S}_k^0(38)}) &= -(-19)^{\mathcal{Z}} + \delta_{k,2}, \\
\\
\mathrm{tr}(T(21)|_{\mathcal{S}_k^0(21)}) &= -2(-21)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(21)|_{\mathcal{S}_k^0(42)}) &= 2(-21)^{\mathcal{Z}} - \delta_{k,2}, \\
\\
\mathrm{tr}(T(22)|_{\mathcal{S}_k^0(11)}) &= (-22)^{\mathcal{Z}} - 3\delta_{k,2}, \\
\mathrm{tr}(T(22)|_{\mathcal{S}_k^0(22)}) &= -(-22)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(22)|_{\mathcal{S}_k^0(33)}) &= -2(-22)^{\mathcal{Z}} + 3\delta_{k,2}, \\
\\
\mathrm{tr}(T(23)|_{\mathcal{S}_k^0(23)}) &= 3(-23)^{\mathcal{Z}} - \delta_{k,2}, \\
\\
\mathrm{tr}(T(26)|_{\mathcal{S}_k^0(13)}) &= 3(-26)^{\mathcal{Z}} - 3\delta_{k,2}, \\
\mathrm{tr}(T(26)|_{\mathcal{S}_k^0(26)}) &= -3(-26)^{\mathcal{Z}} + \delta_{k,2}, \\
\mathrm{tr}(T(26)|_{\mathcal{S}_k^0(39)}) &= -\delta_{k,2}, \\
\\
\mathrm{tr}(T(29)|_{\mathcal{S}_k^0(29)}) &= 3(-29)^{\mathcal{Z}} - \delta_{k,2},
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(\mathrm{T}(30)|_{\mathcal{S}_k^0(15)}) &= -2(-30)^\varkappa + 3\delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(30)|_{\mathcal{S}_k^0(30)}) &= 2(-30)^\varkappa - \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(31)|_{\mathcal{S}_k^0(31)}) &= 3(-31)^\varkappa - \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(33)|_{\mathcal{S}_k^0(33)}) &= -2(-33)^\varkappa + \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(34)|_{\mathcal{S}_k^0(17)}) &= 2(-34)^\varkappa - 3\delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(34)|_{\mathcal{S}_k^0(34)}) &= -2(-34)^\varkappa + \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(35)|_{\mathcal{S}_k^0(35)}) &= -4(-35)^\varkappa + \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(37)|_{\mathcal{S}_k^0(37)}) &= (-37)^\varkappa - \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(38)|_{\mathcal{S}_k^0(19)}) &= 3(-38)^\varkappa - 3\delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(38)|_{\mathcal{S}_k^0(38)}) &= -3(-38)^\varkappa + \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(39)|_{\mathcal{S}_k^0(13)}) &= 4(-39)^\varkappa - 4\delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(39)|_{\mathcal{S}_k^0(26)}) &= 4\delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(39)|_{\mathcal{S}_k^0(39)}) &= -4(-39)^\varkappa + \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(41)|_{\mathcal{S}_k^0(41)}) &= 4(-41)^\varkappa - \delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(42)|_{\mathcal{S}_k^0(14)}) &= -2(-42)^\varkappa + 4\delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(42)|_{\mathcal{S}_k^0(21)}) &= -2(-42)^\varkappa + 3\delta_{k,2}, \\
\mathrm{tr}(\mathrm{T}(42)|_{\mathcal{S}_k^0(42)}) &= 2(-42)^\varkappa - \delta_{k,2},
\end{aligned}$$

7. PROOF OF THEOREM 2

We put $\mathbb{N}(N^\times)$ the set of all divisors of N^\times . For $h, i \in \mathbb{N}(N^\times)$, we define

$$h \triangle i = \prod_{p \in \mathbb{P}(h//i) \cup \mathbb{P}(i//h)} p.$$

Then $\mathbb{N}(N^\times)$ becomes a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\sigma(N^\times)}$ and $\langle \bullet, i \rangle$ becomes a homomorphism $\mathbb{N}(N^\times) \rightarrow \{\pm 1\}$, i.e.

$$\langle h, i \rangle \langle h', i \rangle = \langle h \triangle h', i \rangle.$$

Let $(l, N^\times) = 1$. The orthogonal relation of characters induces

$$\begin{aligned}
\sigma(N^\times) \mathrm{tr}(\mathrm{T}(l)|_{\mathcal{S}_k^0(N; i)}) &= \sum_{j|N^\times} \left(\sum_{h|N^\times} \langle h, i \triangle j \rangle \right) \mathrm{tr}(\mathrm{T}(l)|_{\mathcal{S}_k^0(N; j)}) \\
&= \sum_{h|N^\times} \langle h, i \rangle \sum_{j|N^\times} \langle h, j \rangle \mathrm{tr}(\mathrm{T}(l)|_{\mathcal{S}_k^0(N; j)}).
\end{aligned}$$

Here, we see

$$\begin{aligned}
\langle h, j \rangle \mathrm{tr}(\mathrm{T}(l)|_{\mathcal{S}_k^0(N; j)}) &= \sum_{f \in \mathcal{P}_k(N; j)} a_l(f) \langle h, j \rangle \\
&= \sum_{f \in \mathcal{P}_k(N; j)} a_l(f) \prod_{p \in \mathbb{P}(h)} \frac{a_p(f)}{p^\varkappa} \\
&= h^{-\varkappa} \mathrm{tr}(\mathrm{T}(hl)|_{\mathcal{S}_k^0(N; j)}),
\end{aligned}$$

and thus

$$\sigma(N^\times) \mathrm{tr}(\mathrm{T}(l)|_{\mathcal{S}_k^0(N; i)}) = \sum_{h|N^\times} \langle h, i \rangle h^{-\varkappa} \mathrm{tr}(\mathrm{T}(hl)|_{\mathcal{S}_k^0(N)}).$$

8. EXAMPLES OF DIMENSION OF $\mathcal{S}_k^0(N; i)$

For example, if p is prime and $(p, l) = 1$, then we see

$$\begin{aligned}\dim \mathcal{S}_k^0(p; 1) &= \frac{1}{2} \left(\dim \mathcal{S}_k^0(p) + \frac{1}{p^*} \text{tr}(\text{T}(p)|_{\mathcal{S}_k^0(p)}) \right), \\ \dim \mathcal{S}_k^0(p; p) &= \frac{1}{2} \left(\dim \mathcal{S}_k^0(p) - \frac{1}{p^*} \text{tr}(\text{T}(p)|_{\mathcal{S}_k^0(p)}) \right).\end{aligned}$$

Put

$$d_k(N; i) = \dim \mathcal{S}_k^0(N; i) - \mu(N) \delta_{k,2} \delta_{i,1}.$$

We calculate $d_k(N; i)$ for each $N \leq 42$.

Example 20. For prime N , we write down the following formulas:

k	$d_k(2; 1)$	$d_k(2; 2)$
$2 + 24n$	$1 + n$	n
$4 + 24n$	n	n
$6 + 24n$	n	n
$8 + 24n$	n	$1 + n$
$10 + 24n$	$1 + n$	n
$12 + 24n$	n	n
$14 + 24n$	$1 + n$	$1 + n$
$16 + 24n$	n	$1 + n$
$18 + 24n$	$1 + n$	n
$20 + 24n$	$1 + n$	$1 + n$
$22 + 24n$	$1 + n$	$1 + n$
$24 + 24n$	n	$1 + n$

k	$d_k(3; 1)$	$d_k(3; 3)$	$d_k(5; 1)$	$d_k(5; 5)$	$d_k(11; 1)$	$d_k(11; 11)$
$2 + 12n$	$1 + n$	n	$1 + 2n$	$2n$	$2 + 5n$	$5n$
$4 + 12n$	n	n	$2n$	$1 + 2n$	$5n$	$2 + 5n$
$6 + 12n$	$1 + n$	n	$1 + 2n$	$2n$	$3 + 5n$	$1 + 5n$
$8 + 12n$	n	$1 + n$	$1 + 2n$	$2 + 2n$	$2 + 5n$	$4 + 5n$
$10 + 12n$	$1 + n$	$1 + n$	$2 + 2n$	$1 + 2n$	$5 + 5n$	$3 + 5n$
$12 + 12n$	n	$1 + n$	$1 + 2n$	$2 + 2n$	$3 + 5n$	$5 + 5n$

k	$d_k(17; 1)$	$d_k(17; 17)$	$d_k(23; 1)$	$d_k(23; 23)$
$2 + 12n$	$2 + 8n$	$8n$	$3 + 11n$	$11n$
$4 + 12n$	$1 + 8n$	$3 + 8n$	$1 + 11n$	$4 + 11n$
$6 + 12n$	$4 + 8n$	$2 + 8n$	$6 + 11n$	$3 + 11n$
$8 + 12n$	$4 + 8n$	$6 + 8n$	$5 + 11n$	$8 + 11n$
$10 + 12n$	$7 + 8n$	$5 + 8n$	$10 + 11n$	$7 + 11n$
$12 + 12n$	$6 + 8n$	$8 + 8n$	$8 + 11n$	$11 + 11n$

k	$d_k(29; 1)$	$d_k(29; 29)$	$d_k(41; 1)$	$d_k(41; 41)$
$2 + 12n$	$3 + 14n$	$14n$	$4 + 20n$	$20n$
$4 + 12n$	$2 + 14n$	$5 + 14n$	$3 + 20n$	$7 + 20n$
$6 + 12n$	$7 + 14n$	$4 + 14n$	$10 + 20n$	$6 + 20n$
$8 + 12n$	$7 + 14n$	$10 + 14n$	$10 + 20n$	$14 + 20n$
$10 + 12n$	$12 + 14n$	$9 + 14n$	$17 + 20n$	$13 + 20n$
$12 + 12n$	$11 + 14n$	$14 + 14n$	$16 + 20n$	$20 + 20n$

k	$d_k(7; 1)$	$d_k(7; 7)$	$d_k(13; 1)$	$d_k(13; 13)$	$d_k(19; 1)$	$d_k(19; 19)$
$2 + 4n$	$1 + n$	n	$1 + 2n$	$2n$	$2 + 3n$	$3n$
$4 + 4n$	n	$1 + n$	$1 + 2n$	$2 + 2n$	$1 + 3n$	$3 + 3n$

k	$d_k(31; 1)$	$d_k(31; 31)$	$d_k(37; 1)$	$d_k(37; 37)$
$2 + 4n$	$3 + 5n$	$5n$	$2 + 6n$	$1 + 6n$
$4 + 4n$	$2 + 5n$	$5 + 5n$	$4 + 6n$	$5 + 6n$

Example 21. For square-free composite N , we write down the following formulas:

k	$d_k(6; 1)$	$d_k(6; 2)$	$d_k(6; 3)$	$d_k(6; 6)$
$2 + 24n$	$-1 + n$	n	n	n
$4 + 24n$	n	n	n	$1 + n$
$6 + 24n$	n	n	$1 + n$	n
$8 + 24n$	$1 + n$	n	n	n
$10 + 24n$	n	$1 + n$	n	n
$12 + 24n$	$1 + n$	$1 + n$	n	$1 + n$
$14 + 24n$	n	n	$1 + n$	n
$16 + 24n$	$1 + n$	n	$1 + n$	$1 + n$
$18 + 24n$	n	$1 + n$	$1 + n$	$1 + n$
$20 + 24n$	$1 + n$	$1 + n$	n	$1 + n$
$22 + 24n$	$1 + n$	$1 + n$	$1 + n$	n
$24 + 24n$	$2 + n$	$1 + n$	$1 + n$	$1 + n$

k	$d_k(10; 1)$	$d_k(10; 2 \text{ or } 3 \text{ or } 5)$
$2 + 12n$	$-1 + n$	n
$4 + 12n$	$1 + n$	n
$6 + 12n$	n	$1 + n$
$8 + 12n$	$1 + n$	n
$10 + 12n$	n	$1 + n$
$12 + 12n$	$2 + n$	$1 + n$

k	$d_k(14; 1)$	$d_k(14; 2)$	$d_k(14; 7)$	$d_k(14; 14)$
$2 + 8n$	$-1 + n$	$1 + n$	n	n
$4 + 8n$	$1 + n$	n	n	$1 + n$
$6 + 8n$	n	$1 + n$	$1 + n$	n
$8 + 8n$	$2 + n$	n	$1 + n$	$1 + n$

k	$d_k(15; 1)$	$d_k(15; 3)$	$d_k(15; 5)$	$d_k(15; 15)$
$2 + 12n$	$-1 + 2n$	$1 + 2n$	$2n$	$2n$
$4 + 12n$	$1 + 2n$	$2n$	$2n$	$1 + 2n$
$6 + 12n$	$2n$	$2 + 2n$	$1 + 2n$	$1 + 2n$
$8 + 12n$	$2 + 2n$	$2n$	$1 + 2n$	$1 + 2n$
$10 + 12n$	$1 + 2n$	$2 + 2n$	$2 + 2n$	$1 + 2n$
$12 + 12n$	$3 + 2n$	$1 + 2n$	$2 + 2n$	$2 + 2n$

k	$d_k(21; 1)$	$d_k(21; 3 \text{ or } 7)$	$d_k(21; 7)$
$2 + 4n$	$-1 + n$	n	$1 + n$
$4 + 4n$	$2 + n$	$1 + n$	n

k	$d_k(22; 1)$	$d_k(22; 2 \text{ or } 22)$	$d_k(22; 11)$
$2 + 24n$	$-1 + 5n$	$5n$	$5n$
$4 + 24n$	$1 + 5n$	$1 + 5n$	$5n$
$6 + 24n$	$1 + 5n$	$1 + 5n$	$2 + 5n$
$8 + 24n$	$2 + 5n$	$1 + 5n$	$1 + 5n$
$10 + 24n$	$1 + 5n$	$2 + 5n$	$2 + 5n$
$12 + 24n$	$3 + 5n$	$3 + 5n$	$2 + 5n$
$14 + 24n$	$2 + 5n$	$2 + 5n$	$3 + 5n$
$16 + 24n$	$4 + 5n$	$3 + 5n$	$3 + 5n$
$18 + 24n$	$3 + 5n$	$4 + 5n$	$4 + 5n$
$20 + 24n$	$4 + 5n$	$4 + 5n$	$3 + 5n$
$22 + 24n$	$4 + 5n$	$4 + 5n$	$5 + 5n$
$24 + 24n$	$6 + 5n$	$5 + 5n$	$5 + 5n$

k	$d_k(26; 1)$	$d_k(26; 2 \text{ or } 13)$	$d_k(26; 26)$
$2 + 4n$	$-1 + n$	$1 + n$	n
$4 + 4n$	$2 + n$	n	$1 + n$

k	$d_k(33; 1)$	$d_k(33; 3)$	$d_k(33; 11)$	$d_k(33; 33)$
$2 + 12n$	$-1 + 5n$	$1 + 5n$	$5n$	$5n$
$4 + 12n$	$2 + 5n$	$1 + 5n$	$1 + 5n$	$2 + 5n$
$6 + 12n$	$1 + 5n$	$3 + 5n$	$2 + 5n$	$2 + 5n$
$8 + 12n$	$4 + 5n$	$2 + 5n$	$3 + 5n$	$3 + 5n$
$10 + 12n$	$3 + 5n$	$4 + 5n$	$4 + 5n$	$3 + 5n$
$12 + 12n$	$6 + 5n$	$4 + 5n$	$5 + 5n$	$5 + 5n$

k	$d_k(34; 1)$	$d_k(34; 2 \text{ or } 34)$	$d_k(34; 17)$
$2 + 12n$	$-1 + 4n$	$4n$	$1 + 4n$
$4 + 12n$	$2 + 4n$	$1 + 4n$	$4n$
$6 + 12n$	$1 + 4n$	$2 + 4n$	$3 + 4n$
$8 + 12n$	$3 + 4n$	$2 + 4n$	$1 + 4n$
$10 + 12n$	$2 + 4n$	$3 + 4n$	$4 + 4n$
$12 + 12n$	$5 + 4n$	$4 + 4n$	$3 + 4n$

k	$d_k(35; 1)$	$d_k(35; 5)$	$d_k(35; 7)$	$d_k(35; 35)$
$2 + 4n$	$-1 + 2n$	$1 + 2n$	$2 + 2n$	$2n$
$4 + 4n$	$3 + 2n$	$1 + 2n$	$2n$	$2 + 2n$

k	$d_k(38; 1)$	$d_k(38; 2)$	$d_k(38; 19)$	$d_k(38; 38)$
$2 + 8n$	$-1 + 3n$	$1 + 3n$	$1 + 3n$	$3n$
$4 + 8n$	$2 + 3n$	$1 + 3n$	$3n$	$2 + 3n$
$6 + 8n$	$1 + 3n$	$2 + 3n$	$3 + 3n$	$1 + 3n$
$8 + 8n$	$4 + 3n$	$2 + 3n$	$2 + 3n$	$3 + 3n$

k	$d_k(39; 1)$	$d_k(39; 3)$	$d_k(39; 13)$	$d_k(39; 39)$
$2 + 4n$	$-1 + 2n$	$1 + 2n$	$2 + 2n$	$2n$
$4 + 4n$	$3 + 2n$	$1 + 2n$	$2n$	$2 + 2n$

k	$d_k(30; 1 \text{ or } 10)$	$d_k(30; 3 \text{ or } 6 \text{ or } 15 \text{ or } 30)$	$d_k(30; 2 \text{ or } 5)$
$2 + 12n$	$1 + n$	n	n
$4 + 12n$	n	n	$1 + n$
$6 + 12n$	$1 + n$	n	n
$8 + 12n$	n	$1 + n$	$1 + n$
$10 + 12n$	$1 + n$	$1 + n$	n
$12 + 12n$	n	$1 + n$	$1 + n$

k	$d_k(42; 1 \text{ or } 21)$	$d_k(42; 2 \text{ or } 6 \text{ or } 14 \text{ or } 42)$	$d_k(42; 3 \text{ or } 7)$
$2 + 8n$	$1 + n$	n	n
$4 + 8n$	n	n	$1 + n$
$6 + 8n$	$1 + n$	$1 + n$	n
$8 + 8n$	n	$1 + n$	$1 + n$

Example 22. For not square-free N , we write down the following formulas:

k	$d_k(12; 1)$	$d_k(12; 3)$	$d_k(20; 1)$	$d_k(20; 5)$
$2 + 12n$	n	n	$2n$	$1 + 2n$
$4 + 12n$	$1 + n$	n	$1 + 2n$	$2n$
$6 + 12n$	n	n	$2n$	$1 + 2n$
$8 + 12n$	$1 + n$	$1 + n$	$2 + 2n$	$1 + 2n$
$10 + 12n$	n	$1 + n$	$1 + 2n$	$2 + 2n$
$12 + 12n$	$1 + n$	$1 + n$	$2 + 2n$	$1 + 2n$

k	$d_k(18; 1)$	$d_k(18; 2)$
$2 + 24n$	$5n$	$5n$
$4 + 24n$	$1 + 5n$	$5n$
$6 + 24n$	$1 + 5n$	$2 + 5n$
$8 + 24n$	$1 + 5n$	$1 + 5n$
$10 + 24n$	$2 + 5n$	$2 + 5n$
$12 + 24n$	$3 + 5n$	$2 + 5n$
$14 + 24n$	$2 + 5n$	$3 + 5n$
$16 + 24n$	$3 + 5n$	$3 + 5n$
$18 + 24n$	$4 + 5n$	$4 + 5n$
$20 + 24n$	$4 + 5n$	$3 + 5n$
$22 + 24n$	$4 + 5n$	$5 + 5n$
$24 + 24n$	$5 + 5n$	$5 + 5n$

k	$d_k(24; 1)$	$d_k(24; 3)$	$d_k(28; 1 \text{ or } 7)$	$d_k(40; 1)$	$d_k(40; 5)$
$2 + 4n$	n	$1 + n$	n	$1 + 2n$	$2n$
$4 + 4n$	$1 + n$	n	$1 + n$	$1 + 2n$	$2 + 2n$

9. SOME PRIMITIVE FORMS FOR $N = 14$

9.1. Modular forms. We recall some facts on modular forms, for the next subsection. For a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, we denote by $\mathcal{M}_k(\Gamma)$ the space of all modular forms of weight k with respect to Γ . We put $\mathcal{M}_k(N) = \mathcal{M}_k(\Gamma_0(N))$. Moreover, we define

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid a \equiv 1 \pmod{N} \right\},$$

and for each Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$

$$\mathcal{M}_k(N, \chi) = \left\{ f \in \mathcal{M}_k(\Gamma_1(N)) \mid (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \chi(d)f \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.$$

Put

$$C_2 = 1 + 24 \sum_{n=1}^{\infty} \left(\sum_{d|n} \mathbf{1}_2(d) d \right) q^n \in \mathcal{M}_2(2),$$

$$F_7 = 1 + 2 \sum_{n=1}^{\infty} \left(\sum_{d|n} \rho_7(d) \right) q^n \in \mathcal{M}_1(7, \rho_7),$$

where $\mathbf{1}_N : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \{1\}$ and $\mathbf{1}_7 \neq \rho_7 : (\mathbb{Z}/7\mathbb{Z})^\times \rightarrow \{1, -1\}$. See [3, §4] or [?, §5.3] for details.

We regard $\mathcal{M}_k(\Gamma_1(N)) \subset \mathbb{C}[[q]]$ via the Fourier expansion.

Lemma 23. *We see*

$$\mathcal{M}_2(14) \cap \mathbb{C}[[q]]q^4 = \{0\}.$$

Proof. Put

$$\alpha = \frac{1}{2}(F_7 - F_7^{(2)}) \in \mathcal{M}_1(14, \rho_7) \cap \mathbb{C}[[q]]q,$$

$$\gamma = \frac{1}{8}(F_7^2 - 2F_7F_7^{(2)} + C_2^{(7)}) \in \mathcal{M}_2(14) \cap \mathbb{C}[[q]]q^3,$$

where $f^{(h)}(q) = f(q^h)$. Since $\dim \mathcal{M}_2(14) = 4$ (cf. [3, Theorem 3.5.1]), we see

$$\mathcal{M}_2(14) = \mathbb{C}F_7^2 \oplus \mathbb{C}F_7\alpha \oplus \mathbb{C}\alpha^2 \oplus \mathbb{C}\gamma,$$

and we easily get the assertion. \square

Note that a weaker result

$$\mathcal{M}_2(14) \cap \mathbb{C}[[q]]q^5 = \{0\}$$

can be obtained directly from a result of Sturm [9] and $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(14)] = 24$.

9.2. Primitive forms. We represent all primitive forms of weight 2, 4, in terms of C_2 and F_7 . First, put

$$\Delta = \frac{1}{2}(F_7 - F_7^{(2)})(2F_7^{(2)} - F_7).$$

Example 24. *We see*

$$\mathcal{P}_2(14; 2) = \{\Delta\}.$$

Proof. First, we easily see

$$\Delta \in \mathcal{M}_2(14) \cap (q - q^2 - 2q^3 + \mathbb{C}[[q]]q^4).$$

Note $\mathcal{S}_2(14) = \mathcal{S}_2(14; 2)$ and $\dim \mathcal{S}_2(14) = 1$ by Example 20. Let $\mathcal{P}_2(14; 2) = \{f\}$, then we see $a_1(f) = 1$ and

$$a_2(f) = -2^0 = -1,$$

$$a_3(f) = \mathrm{tr}(\mathrm{T}(3)|_{\mathcal{S}_2(14)}) = -2.$$

Thus, we see

$$f - \Delta \in \mathcal{M}_2(14) \cap \mathbb{C}[[q]]q^4 = \{0\}$$

i.e. $f = \Delta$, and get the assertion. \square

We remark that Δ may be represented as a multiplicative η -product (cf. [2]).

Lemma 25. *For $k \geq 0$, we get*

$$\mathcal{S}_{k+2}(14) = \Delta \mathcal{M}_k(14).$$

Proof. The assertion follows from $\mathcal{S}_{k+2}(14) \supset \Delta \mathcal{M}_k(14)$ and

$$\dim \mathcal{S}_{k+2}(14) = \dim \mathcal{M}_k(14) = \dim(\Delta \mathcal{M}_k(14)).$$

□

Example 26. *We see*

$$\mathcal{P}_4(14; 1) = \left\{ \frac{1}{8} \Delta(C_2 + 7C_2^{(7)}) \right\},$$

$$\mathcal{P}_4(14; 14) = \left\{ \frac{1}{2} \Delta(3F_7^2 - 7F_7F_7^{(2)} + 6F_7^{(2)^2}) \right\}.$$

Proof. At first, we see

$$\mathcal{S}_4(14) \cap \mathbb{C}[[q]]q^5 = \Delta(\mathcal{M}_2(14) \cap \mathbb{C}[[q]]q^4) = \{0\}$$

by Lemma 25 and 23. Note that $\dim \mathcal{S}_4(14; 1) = 1$ by Example 20. Let $\mathcal{P}_4(14; 1) = \{f\}$, then we see $a_1(f) = 1$, $a_2(f) = 2$ and $a_4(f) = a_2(f)^2 = 4$. In addition, we see

$$a_3(f) = \frac{1}{4}(6 - 10 - 10 + 6) = -2$$

by Theorem 2 and

$$\begin{aligned} \text{tr}(T(3)|_{\mathcal{S}_4(14)}) &= 6, \\ \frac{1}{2} \text{tr}(T(6)|_{\mathcal{S}_4(14)}) &= -10, \\ \frac{1}{7} \text{tr}(T(21)|_{\mathcal{S}_4(14)}) &= -10, \\ \frac{1}{14} \text{tr}(T(42)|_{\mathcal{S}_4(14)}) &= 6. \end{aligned}$$

Thus, we get

$$f - \frac{1}{8} \Delta(C_2 + 7C_2^{(7)}) \in \mathcal{S}_4(14) \cap \mathbb{C}[[q]]q^5 = \{0\}.$$

We get the second assertion in a similar way. □

For many other examples for $N = 1, 2, 3, 4, 6, 8, 9$, see [10].

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